

THE LATTICE THEORY OF POST ALGEBRAS

BY

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1. Introduction. The algebra corresponding to the n -valued logic described by E. L. Post [7] in 1921 was first developed by P. C. Rosenbloom [9] in 1942. Rosenbloom's axioms are based on a minimum of undefined operations and are therefore quite complicated. This complexity also hinders his development of the theory. In this paper, a set of axioms for Post algebras is presented which makes use of a greater number of operations, as well as certain constants. These operations C_0, \dots, C_{n-1} are generalized complementation operators, where n is the order of the Post algebra. The axioms, given in terms of these operators, are very simple. In addition, the simplicity of the operations makes a large part of the theory much more transparent. Another striking feature of the development is the role played by the underlying Boolean algebra of the Post algebra. The existence of this Boolean algebra has been known for a long time, but this fact has not been as fully exploited as in this approach. It will be shown, for example, that the representation theory for Post algebras follows immediately from the corresponding theory for Boolean algebras. No further use of the Axiom of Choice is needed. In addition many properties of a Post algebra, such as completeness, infinite distributivity, and the atomistic property, are fully mirrored by the corresponding properties for the underlying Boolean algebra.

The notation is explained in §2. §3 presents the axioms, and various theorems and remarks concerning the arithmetic and structure of the algebra. §4 discusses Post functions and their reduction to a given form. Examples are given in §5. The representation theory is described in §6, and §7 discusses completeness properties of infinite Post algebras.

2. Notation. The usual lattice notation is employed. The supremum of x and y is denoted by $x \vee y$, and the infimum of x and y is denoted by $x \wedge y$, or more briefly, by xy . The symbols $\vee x_i$ and $\wedge x_i$ denote, respectively, the supremum and infimum of the x_i over a specified index set. The symbols $\vee_{L_i} x_i$ and $\wedge_{L_i} x_i$ emphasize that the supremum or infimum is taken in the lattice L . If x has a complement, it is denoted by $\{x\}^-$ or, if convenient, by \bar{x} .

3. Formulation. Let n be a fixed integer satisfying $n \geq 2$. Let L be a distributive lattice with zero 0 and unit u , and satisfying the following conditions:

AXIOM 1. For every element $x \in L$ there exist n elements $C_0(x), C_1(x), \dots, C_{n-1}(x)$ which are pairwise disjoint and whose supremum is u ; that is, $C_i(x)C_j(x) = 0$ for $i \neq j$ and $\bigvee_{i=0}^{n-1} C_i(x) = u$.

Presented to the Society, June 20, 1959; received by the editors April 24, 1959.

AXIOM 2. There exist n fixed elements of L , denoted $0 = e_0, e_1, \dots, e_{n-2}, e_{n-1} = u$ with the properties:

2a. The elements form a chain, with $e_{i-1} \leq e_i$ for $1 \leq i \leq n-1$.

2b. If $x \in L$ and $xe_1 = 0$, then $x = 0$.

2c. If $x \in L$ and, for some i , $x \vee e_{i-1} = e_i$, then $x = e_i$.

AXIOM 3. For every $x \in L$, $x = \bigvee_{i=0}^{n-1} e_i C_i(x)$; that is,

$$x = e_1 C_1(x) \vee e_2 C_2(x) \vee \dots \vee e_{n-2} C_{n-2}(x) \vee C_{n-1}(x).$$

It will be shown in §6 that L is a Post algebra and, conversely, every Post algebra is such a lattice. The remaining work in this section is an exposition of fundamental consequences.

LEMMA 1. If $x \in L$ and $xe_i = 0$ for some i in the range $1 \leq i \leq n-1$, then $x = 0$.

Proof. If $xe_i = 0$, then $(xe_i)e_1 = 0$. Hence $xe_1 = 0$ by Axiom 2a and $x = 0$ by Axiom 2b.

LEMMA 2. If $x \in L$ and if $x \vee e_i = e_j$ for some i and j , with $i < j$, then $x = e_j$.

Proof. If $x \vee e_i = e_j$, then $x \vee e_i \vee e_{j-1} = e_j \vee e_{j-1}$, and $x \vee e_{j-1} = e_j$ by Axiom 2a. Hence $x = e_j$ by Axiom 2c.

It is well known that the complemented elements of a distributive lattice form a sublattice which is a Boolean algebra. This leads to the following definition.

DEFINITION 1. Let B be the Boolean algebra of complemented elements of L .

THEOREM 1. If $x \in L$, then $x \in B$ if and only if $x = C_i(y)$ for some i and some $y \in L$.

Proof. If $x = C_i(y)$, then $\bar{x} = \bigvee_{j \neq i} C_j(y)$ by Axiom 1. On the other hand, suppose x has a complement \bar{x} . Then $x \leq e_{n-2} \vee C_{n-1}(x)$ from Axiom 3, so $e_{n-2} \vee \bar{x} \vee C_{n-1}(x) = u$. Hence $\bar{x} \vee C_{n-1}(x) = u$, by Lemma 2, and therefore $x \leq C_{n-1}(x)$. Since $x \geq C_{n-1}(x)$ by Axiom 3, it follows that $x = C_{n-1}(x)$.

LEMMA 3. If $b \in B$ and $be_i = be_j$ for some i and j with $i < j$, then $b = 0$.

Proof. Using Axiom 2a, $e_j = e_j \bar{b} \vee e_j b = e_j \bar{b} \vee e_i b \leq e_j \bar{b} \vee e_i \leq e_j$, so that $\bar{b}e_j \vee e_i = e_j$. Hence $\bar{b}e_j = e_j$ by Lemma 2. Thus $be_j = b(\bar{b}e_j) = 0$, and, by Lemma 1, $b = 0$.

THEOREM 2. (UNIQUENESS THEOREM). The $C_i(x)$ are unique. That is, for any given $x \in L$, there is only one sequence of elements $C_0(x), C_1(x), \dots, C_{n-1}(x)$ satisfying Axiom 1 and Axiom 3.

Proof. Suppose there is another such sequence, say $C'_0(x), C'_1(x), \dots, C'_{n-1}(x)$. By Axiom 3, $\bigvee_{k=0}^{n-1} e_k C_k(x) = \bigvee_{k=0}^{n-1} e_k C'_k(x)$. Thus if $i \neq j$, then $C_i(x)C'_j(x) \vee \bigvee_{k=0}^{n-1} e_k C_k(x) = C_i(x)C'_j(x) \vee \bigvee_{k=0}^{n-1} e_k C'_k(x)$, and, by Axiom 1, $e_i C_i(x)C'_j(x) = e_j C_i(x)C'_j(x)$. Since $C_i(x)C'_j(x) \in B$, Lemma 3 shows that

$C_i(x)C_j'(x)=0$. The result now follows from Axiom 1, for using the fact that $u = \bigvee_{k=0}^{n-1} C_k(x)$, for every j , $C_j'(x) = C_j'(x) \bigvee_{k=0}^{n-1} C_k(x)$, and $C_j'(x) = C_j'(x)C_j(x)$. Likewise $u = \bigvee_{k=0}^{n-1} C_k'(x)$, so that $C_j(x) = C_j(x) \bigvee_{k=0}^{n-1} C_k'(x) = C_j(x)C_j'(x)$. Therefore $C_j(x) = C_j'(x)$ for each j , and the proof is complete.

THEOREM 3. *For each i in the range $0 \leq i \leq n-1$,*

$$C_{n-1}(C_i(x)) = C_i(x),$$

$$C_j(C_i(x)) = 0, \quad \text{for } 0 < j < n-1,$$

and

$$C_0(C_i(x)) = \bigvee_{k \neq i} C_k(x) = \{C_i(x)\}^-.$$

Proof. For given i , the n elements $\bigvee_{k \neq i} C_k(x)$, 0 , $0, \dots, 0$, $C_i(x)$ are pairwise disjoint and their supremum is u , so that Axiom 1 is satisfied. Axiom 3 is satisfied since $C_i(x) = e_1 0 \vee \dots \vee e_{n-2} 0 \vee C_i(x)$. Hence the result follows by the uniqueness theorem.

COROLLARY. *If $b \in B$, then $C_0(b) = \bar{b}$, $C_j(b) = 0$ for $0 < j < n-1$, and $C_{n-1}(b) = b$.*

Proof. This follows from Theorem 1.

THEOREM 4. *If $i \neq j$, then $C_i(e_j) = 0$, and $C_i(e_i) = u$. The elements e_i , $i = 0, 1, \dots, n-1$ are distinct and unique.*

Proof. The first part of the theorem is again an immediate consequence of the uniqueness theorem. If $i \neq j$ and $e_i = e_j$, then $C_i(e_i) = u$ and $C_i(e_j) = C_j(e_j) = u$, contradicting Axiom 1. If there is another sequence of elements $0, e'_1, \dots, e'_{n-2}, u$ satisfying Axiom 2 and Axiom 3, then for every $x \in L$ $x = \bigvee_{k=0}^{n-1} e'_k C_k(x)$, so that for $x = e_i$, $i = 1, 2, \dots, n-2$, $e_i = e'_i$. Thus the elements e_i , $i = 0, 1, \dots, n-1$ are both unique and distinct.

THEOREM 5. *The lattice L is pseudo-complemented; that is, if $x \in L$, there exists $x^* \in L$ such that $xy = 0$ if and only if $y \leq x^*$.*

Proof. It is clear that $x^* = C_0(x)$, for by Axiom 3 and Axiom 1 $xC_0(x) = 0$, and $xy = 0$ if $y \leq C_0(x)$. Conversely, if $xy = 0$, then $e_k C_k(x)y = 0$ for each k , since $y \bigvee_{k=1}^{n-1} e_k C_k(x) = 0$. Hence $C_k(x)y = 0$ for each $k \geq 1$ by Lemma 1. Thus $y \bigvee_{k=1}^{n-1} C_k(x) = y \{C_0(x)\}^- = 0$, and $y \leq C_0(x)$.

THEOREM 6. *If $b_i \in B$ and $x = \bigvee_{i=1}^{n-1} e_i b_i$, then $x = \bigvee_{i=1}^{n-1} e_i (\bigvee_{j=i}^{n-1} b_j)$, and*

$$C_0(x) = \bigwedge_{j=1}^{n-1} b_j, \quad C_i(x) = b_i \bigwedge_{j=i+1}^{n-1} b_j \quad \text{for } i = 1, \dots, n-2,$$

and $C_{n-1}(x) = b_{n-1}$.

Proof. If $x = \bigvee_{j=1}^{n-1} e_j b_j$, then

$$x = \bigvee_{j=1}^{n-1} \left(\bigvee_{i=1}^j e_i \right) b_j = \bigvee_{i=1}^{n-1} e_i \left(\bigvee_{j=i}^{n-1} b_j \right).$$

It is clear that

$$\bigvee_{i=k}^{n-2} \left(b_i \left(\bigwedge_{j=i+1}^{n-1} b_j \right) \right) \vee b_{n-1} = \bigvee_{i=k}^{n-1} b_i \quad \text{for } 0 < k < n-1.$$

For brevity, let $c_i = b_i (\bigwedge_{j=i+1}^{n-1} b_j)$, $0 < i < n-1$, and set $c_{n-1} = b_{n-1}$. Then

$$x = \bigvee_{i=1}^{n-1} e_i \left(\bigvee_{j=i}^{n-1} b_j \right) = \bigvee_{i=1}^{n-1} e_i \left(\bigvee_{j=i}^{n-1} c_j \right) = \bigvee_{j=1}^{n-1} c_j \left(\bigvee_{i=1}^j e_i \right) = \bigvee_{j=1}^{n-1} e_j c_j.$$

Since the n elements $\bigwedge_{i=1}^{n-1} b_i$, c_1, \dots, c_{n-1} are pairwise disjoint and their supremum is u , the uniqueness theorem yields the final result.

The following definition and consequent theorems are preliminary to the work of §7. They are also used to give simple proofs of Theorem 10 and Theorem 12.

DEFINITION 2. Let $D_i(x) = \bigvee_{j=i}^{n-1} C_j(x)$, $i=1, 2, \dots, n-1$.

Clearly $D_i(x) \geq D_j(x) \in B$ for all $x \in L$ and all i and j in the range $1 \leq i \leq j \leq n-1$. Axiom 3 and Theorem 6 show that $x = \bigvee_{i=1}^{n-1} e_i D_i(x)$. The following theorem is another consequence of Theorem 6.

THEOREM 7. If $b_i \geq b_j \in B$ for all i and j in the range $1 \leq i \leq j \leq n-1$, then $b_i = D_i(x)$, where $x = \bigvee_{i=1}^{n-1} e_i b_i$.

Proof. Using Theorem 6, $D_k(x) = \bigvee_{j=k}^{n-1} C_j(x) = \bigvee_{j=k}^{n-1} b_j = b_k$.

THEOREM 8. For each $i=1, 2, \dots, n-1$ and for every $x \in L$ and $y \in L$, $D_i(x \vee y) = D_i(x) \vee D_i(y)$.

Proof. Since $D_i(x) \geq D_j(x) \in B$ and $D_i(y) \geq D_j(y) \in B$ for all i and j in the range $1 \leq i \leq j \leq n-1$, $D_i(x) \vee D_i(y) \geq D_j(x) \vee D_j(y) \in B$. Since

$$x \vee y = \bigvee_{i=1}^{n-1} e_i D_i(x) \vee \bigvee_{i=1}^{n-1} e_i D_i(y) = \bigvee_{i=1}^{n-1} e_i (D_i(x) \vee D_i(y)),$$

the result follows from Theorem 7.

THEOREM 9. A necessary and sufficient condition that $x \leq y$ is that $D_i(x) \leq D_i(y)$ for each $i=1, \dots, n-1$.

Proof. If $x \leq y$, then $x \vee y = y$ and $D_i(x) \vee D_i(y) = D_i(y)$ by Theorem 8; that is, $D_i(x) \leq D_i(y)$ for each $i=1, 2, \dots, n-1$. Conversely, if $D_i(x) \leq D_i(y)$ for each $i=1, 2, \dots, n-1$, then

$$x = \bigvee_{i=1}^{n-1} e_i D_i(x) \leq \bigvee_{i=1}^{n-1} e_i D_i(y) = y.$$

THEOREM 10 (DUALITY THEOREM). *The lattice L is dually isomorphic with itself under the mapping $\beta(x) = \bigvee_{i=1}^{n-1} e_i \{D_{n-i}(x)\}^-$. If L' denotes the dual lattice and C'_i, e'_i are the operators and elements of Axiom 1 and Axiom 2 for L' , while D'_i are the operators of Definition 2 for L' , then*

$$(1) \quad e'_i = e_{n-i-1},$$

$$(2) \quad C'_i(x) = \{C_{n-i-1}(x)\}^-,$$

and

$$(3) \quad D'_i(x) = D_{n-i}(x).$$

Proof. Since $\{D_{n-i}(x)\}^- \geq \{D_{n-j}(x)\}^- \in B$ for all i and j in the range $1 \leq i \leq j \leq n-1$, Theorem 7 yields

$$(4) \quad D_i(\beta(x)) = \{D_{n-i}(x)\}^-.$$

This shows that

$$(5) \quad \beta(\beta(x)) = x,$$

and it follows from Theorem 9 that $x \leq y$ if and only if $\beta(x) \geq \beta(y)$. Hence the dual lattice L' is isomorphic with the lattice L , and in particular L' is a distributive lattice satisfying Axioms 1, 2, and 3.

Since $\{D_{n-i}(x)\}^- = \bigvee_{j=0}^{n-i-1} C_j(x)$ and

$$\bigvee_{i=1}^{n-1} e_i \left(\bigvee_{j=0}^{n-i-1} C_j(x) \right) = \bigvee_{j=0}^{n-2} C_j(x) \left(\bigvee_{i=1}^{n-j-1} e_i \right) = \bigvee_{j=0}^{n-2} C_j(x) e_{n-j-1}$$

it follows that

$$(6) \quad \beta(x) = \bigvee_{i=1}^{n-1} e_i C_{n-i-1}(x).$$

Hence $e'_i = \beta(e_i) = e_{n-i-1}$ by Theorem 4, and (1) is proven. Then by the uniqueness theorem $C_i(\beta(x)) = C_{n-i-1}(x)$. If $b \in B$, then by the Corollary of Theorem 3 $C_i(\beta(b)) = C_{n-i-1}(b) = 0$ for $1 \leq i \leq n-2$ and $C_{n-1}(\beta(b)) = C_0(b) = \bar{b}$. Thus (6) yields $\beta(b) = \bar{b}$. Note also that the set of complemented elements of L' is identical with the set B of complemented elements of L . Now $\beta(C_i(x)) = C'_i(\beta(x))$ by the isomorphism, so that $C'_i(x) = \beta(C_i(\beta(x))) = \{C_i(\beta(x))\}^- = \{C_{n-i-1}(x)\}^-$ for $i=0, 1, \dots, n-1$, and (2) is proven. Similarly, using (4), $D'_i(x) = \beta(D_i(\beta(x))) = \{D_i(\beta(x))\}^- = D_{n-i}(x)$, and this completes the proof.

Since $\beta(x \vee y) = \beta(x)\beta(y)$ and $\beta(xy) = \beta(x) \vee \beta(y)$, it is now easy to obtain the dual form of Axiom 3. In particular, $x = \beta(\beta(x)) = \beta(\bigvee_{i=1}^{n-1} e_i C_{n-i-1}(x))$, so that

$$x = \bigwedge_{i=1}^{n-1} (\beta(e_i) \vee \beta(C_{n-i-1}(x))) = \bigwedge_{i=1}^{n-1} (e_{n-i-1} \vee \{C_{n-i-1}(x)\}^-);$$

that is, $x = \{C_0(x)\}^- (e_1 \vee \{C_1(x)\}^-) \cdots (e_{n-2} \vee \{C_{n-2}(x)\}^-)$.

THEOREM 11. *For each $i=1, 2, \dots, n-1$ and for every $x \in L$ and $y \in L$, $D_i(xy) = D_i(x)D_i(y)$.*

Proof. This is an easy consequence of Theorem 8 and the duality theorem. Specifically, for each $i=1, 2, \dots, n-1$, by Theorem 8, $D_{n-i}(\beta(x) \vee \beta(y)) = D_{n-i}(\beta(x)) \vee D_{n-i}(\beta(y))$. That is, $D_{n-i}(\beta(xy)) = D_{n-i}(\beta(x)) \vee D_{n-i}(\beta(y))$, so that by (4) of Theorem 10, $\{D_i(xy)\}^- = \{D_i(x)\}^- \vee \{D_i(y)\}^-$, and so $D_i(xy) = D_i(x)D_i(y)$.

THEOREM 12. *The following identities are valid for each $i=0, 1, \dots, n-1$.*

$$(1) \quad C_i(x \vee y) = C_i(x) \bigvee_{j=0}^i C_j(y) \vee C_i(y) \bigvee_{j=0}^i C_j(x).$$

$$(2) \quad C_i(xy) = C_i(x) \bigvee_{j=i}^{n-1} C_j(y) \vee C_i(y) \bigvee_{j=i}^{n-1} C_j(x).$$

Proof. These identities are obtained easily through the use of the preceding theorems. Specifically,

$$\begin{aligned} C_i(x \vee y) &= D_i(x \vee y) \{D_{i+1}(x \vee y)\}^- \\ &= (D_i(x) \vee D_i(y)) \{D_{i+1}(x)\}^- \{D_{i+1}(y)\}^- \\ &= C_i(x) \{D_{i+1}(y)\}^- \vee C_i(y) \{D_{i+1}(x)\}^- \\ &= C_i(x) \bigvee_{j=0}^i C_j(y) \vee C_i(y) \bigvee_{j=0}^i C_j(x). \end{aligned}$$

Similarly,

$$\begin{aligned} C_i(xy) &= D_i(xy) \{D_{i+1}(xy)\}^- \\ &= D_i(x)D_i(y) (\{D_{i+1}(x)\}^- \vee \{D_{i+1}(y)\}^-) \\ &= C_i(x)D_i(y) \vee C_i(y)D_i(x) \\ &= C_i(x) \bigvee_{j=i}^{n-1} C_j(y) \vee C_i(y) \bigvee_{j=i}^{n-1} C_j(x). \end{aligned}$$

A more general form of this theorem is given by Theorem 25 in §7.

4. Post functions. The reduction of a Post function to a given form may well be a ponderous chore. It should be noted that the work of this section on canonical expansions and simplifications of Post functions is of an extremely facile nature.

DEFINITION 3. A Post function of m variables is a function which can be

obtained from the constant functions $E_i(x_1, \dots, x_m) = e_i$ and identity functions $I_j(x_1, \dots, x_m) = x_j$ by a finite number of the operations \vee , \wedge , C_0 , C_1, \dots, C_{n-1} .

THEOREM 13. *If f is a Post function of m variables x_1, \dots, x_m , then*

$$f(x_1, \dots, x_m) = \bigvee_{0 \leq i_j \leq n-1} f(e_{i_1}, \dots, e_{i_m}) C_{i_1}(x_1) \cdots C_{i_m}(x_m).$$

Proof. There are n^m terms of the form $C_{i_1}(x_1) C_{i_2}(x_2) \cdots C_{i_m}(x_m)$, and these are called the fundamentals of the m variables. It follows from Axiom 1 that

(1) distinct fundamentals are disjoint, and

(2) the supremum of all the fundamentals is u ; in fact, the infimum of two distinct fundamentals must include an infimum of the form $C_i(x_j) C_k(x_j) = 0$, where $i \neq k$ for some j , and

$$\bigvee_{0 \leq i_j \leq n-1} C_{i_1}(x_1) \cdots C_{i_m}(x_m) = \bigwedge_{j=1}^m \left(\bigvee_{i_j=0}^{n-1} C_{i_j}(x_j) \right) = \bigwedge_{j=1}^m (u) = u.$$

It follows from (2) that the theorem is true for the constant functions E_i , and using Axiom 3, for the identity functions I_j . It is clear that if f and g satisfy the theorem, then $f \vee g$ satisfies the theorem. Using (1), it is also clear that fg satisfies the theorem.

Now suppose f is a Post function satisfying the theorem. Each of the terms $f(e_{i_1}, \dots, e_{i_m})$ is equal to one of the e_j , $0 \leq j \leq n-1$, by Axiom 2a and Theorem 4. Therefore

$$f(x_1, \dots, x_m) = \bigvee_{k=0}^{n-1} e_k T_k,$$

where $T_k = \bigvee C_{i_1}(x_1) \cdots C_{i_m}(x_m)$, the supremum being extended over all (i_1, \dots, i_m) for which $f(e_{i_1}, \dots, e_{i_m}) = e_k$. Using (1) and (2), the uniqueness theorem shows that $C_k(f(x_1, \dots, x_m)) = T_k$. But

$$\bigvee_{0 \leq i_j \leq n-1} C_k(f(e_{i_1}, \dots, e_{i_m})) C_{i_1}(x_1) \cdots C_{i_m}(x_m) = T_k,$$

by Theorem 4. Hence $C_k(f(x_1, \dots, x_m))$ satisfies the theorem. This completes the proof.

THEOREM 14. *If f is a Post function of m variables, then*

$$f(x_1, \dots, x_m) = \bigwedge_{0 \leq i_j \leq n-1} (f(e_{i_1}, \dots, e_{i_m}) \vee \{C_{i_1}(x_1)\}^- \vee \cdots \vee \{C_{i_m}(x_m)\}^-).$$

Proof. The statement follows from the preceding theorem and the duality theorem.

The n^m terms of the form $\{C_{i_1}(x_1)\}^- \vee \cdots \vee \{C_{i_m}(x_m)\}^-$ are called the dual fundamentals. The infimum of all the dual fundamentals is 0, and the

supremum of two distinct dual fundamentals is u —these properties following from the duality theorem applied to (1) and (2) of Theorem 13.

The canonical forms provide a starting point for the simplification of Post functions. The simplification of such functions is greatly aided by contemporary work on the analogous problem in Boolean algebra. In particular, methods described by W. V. Quine [8], the Harvard Computation Laboratory [2], M. Karnaugh [4], and R. H. Urbano and R. K. Mueller [11] can all be easily modified to apply to the more general case of simplification of the lattice theoretic Post functions defined above.

The simplification and manipulation of these functions is further eased by the fact that the fundamentals and dual fundamentals are composed of elements belonging to the Boolean algebra B . Such elements are therefore subject to familiar identities. As a specific illustration, consider the case $n=3$ and the simplifications of the function

$$\begin{aligned} f(x, y) &= C_0(x) \vee C_1(x)C_2(y) \vee C_2(x)C_2(y) \vee e_1C_1(y) \\ &= C_0(x) \vee \{C_0(x)\}^c C_2(y) \vee e_1C_1(y) \\ &= C_0(x) \vee C_2(y) \vee e_1C_1(y) \\ &= C_0(x) \vee y. \end{aligned}$$

Let f be a function in canonical form, such as $f = \bigvee_{i=0}^{n-1} e_i C_i(f)$, where each $C_i(f)$ is expressed as a supremum of fundamentals. It is possible, then, to simplify f by inspection. More systematically, each $C_i(f)$ can be simplified by the modified form of one of the methods mentioned above. The resulting form for f then provides a simplification in terms of the variables e_i and $C_i(x_j)$, $i=0, \dots, n-1$ and $j=1, \dots, m$. If a simplification is required which allows presence of the variables x_j , $j=1, \dots, m$, then this form must be further manipulated, with Axiom 3, to provide a final simplification.

The exact modifications required for each of the above methods are obvious, and need not be elaborated here. The Boolean fundamentals are replaced with the fundamentals defined herein, and the work in general is effected to the base n instead of the base 2.

In Boolean algebra the above methods achieve varying degrees of success, and their relative merits in Post algebra are approximately the same. The "visual recognition" advantage of Karnaugh's map method is emphasized if the final form is to be expressed utilizing the variables x_1, \dots, x_m . The topological method such as outlined by Urbano and Mueller has aroused some recent interest [10]. The figure in the following section illustrates the cube associated with simplifications of Post functions in three variables.

5. Examples. If p is a fixed prime, let R be a p -ring with unit. The ring R is a commutative ring of characteristic p such that $x^p = x$ for every $x \in R$. A ring theoretic function $f(x, y)$ in R is determined by the p^2 values of $f(i, j)$, where $i=0, 1, \dots, p-1$ and $j=0, 1, \dots, p-1$. Let $M(x, y)$ and $m(x, y)$ be those ring theoretic functions for which $M(i, j) = \max(i, j)$ and $m(i, j)$

$=\min(i, j)$, where $i=0, 1, \dots, p-1$ and $j=0, 1, \dots, p-1$. Thus, for example, in the case $p=3$,

$$M(x, y) = x + y + x^2 \cdot y + 2 \cdot x \cdot y + x^2 \cdot y^2 + x \cdot y^2,$$

$$m(x, y) = 2 \cdot x^2 \cdot y + x \cdot y + 2 \cdot x^2 \cdot y^2 + 2 \cdot x \cdot y^2,$$

where the operations involved are the ring operations of addition and multiplication.

It is now a straightforward task to show that if $n=p$, then R forms a lattice L , where $x \vee y = M(x, y)$, $xy = m(x, y)$, $e_i = i$ and $C_i(x) = \prod_{j \neq p-i} (x+j)$, $i=0, 1, \dots, p-1$, the lattice theoretic functions on the left being given in terms of the ring theoretic functions on the right.

Another important example is that of a lattice of n -valued functions which assume the values $0, 1, \dots, n-1$ on a set S , ordered in the usual way, and containing the n constant functions e_i of value i and for each f the n functions $C_i(f)$, $i=0, \dots, n-1$, where $C_i(f)$ is that function which has the value $n-1$ on those points of S where f has the value i , and the value 0 on the other points of S . It is easily verified that Axioms 1, 2, and 3 are satisfied.

The figure on the following page is a Hasse diagram of the free 3-ring with one generator. The points are identified according to the ordering noted above. It is clear that this is also the lattice of all 3-valued functions on a three point set. The eight darkened points in the diagram are the points of B .

6. Representation theory. The work of this section shows, among other things, that the lattice L is equivalent to a Post algebra of order n . This equivalence can of course be proved by tedious calculations which do not make use of the representation theorems.

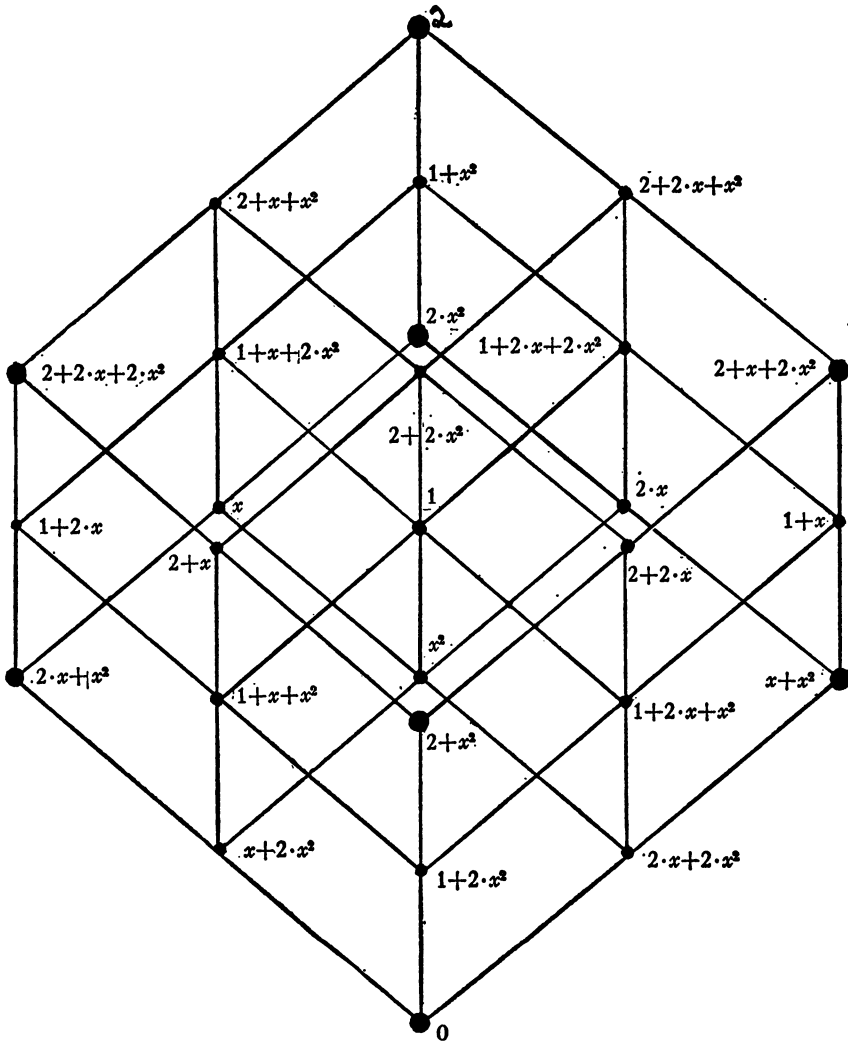
DEFINITION 4. If B is represented as a Boolean algebra of subsets of a set S , where the image of $b \in B$ is given by $\lambda(b) \subset S$, then an n -valued B -measurable function on S is an n -valued function on S with the property that for each $i=0, 1, \dots, n-1$ there exists $b_i \in B$ such that $f^{-1}(i) = \lambda(b_i)$.

THEOREM 15. *If B is represented as a Boolean algebra of a certain class of subsets of a set S , then L is isomorphic with the set of all n -valued B -measurable functions on S .*

Proof. If the image of $b \in B$ is given by $\lambda(b) \subset S$, then for $x \in L$, let x correspond to the function f such that $f=i$ on $\lambda(C_i(x))$ for each $i=0, 1, \dots, n-1$. This correspondence is one-to-one, for if $x \neq y$, then $C_i(x) \neq C_i(y)$ for some i , and $\lambda(C_i(x)) \neq \lambda(C_i(y))$.

Now let f be the function corresponding to $x \in L$ and let g be the function corresponding to $y \in L$. Suppose $i \leq j$ and $P \in \lambda(C_i(x))$, $P \in \lambda(C_j(y))$. Then $P \in \lambda(C_j(y)C_i(x))$, and therefore $P \in \lambda(C_j(x \vee y))$ by Theorem 12. Hence $x \vee y$ corresponds to $f \vee g \equiv \max(f(P), g(P))$ for each $P \in S$.

Finally, the correspondence is onto the set of all n -valued B -measurable functions on S . For let f be such a function. Then there exists $b_i \in B$ such that $\lambda(b_i) = f^{-1}(i)$ for each $i=0, 1, \dots, n-1$. Since $\lambda(b_i) \cap \lambda(b_j) = \emptyset$, the empty



set, for $i \neq j$, and $\bigcup_{i=0}^{n-1} \lambda(b_i) = S$, $b_i b_j = 0$ for $i \neq j$ and $\bigvee_{i=0}^{n-1} b_i = u$. Let $x = \bigvee_{i=0}^{n-1} e_i b_i \in L$. By the uniqueness theorem $C_i(x) = b_i$ for $i = 0, 1, \dots, n-1$, so that f is the function corresponding to x .

THEOREM 16. *The lattice L is isomorphic with the set of all continuous n -valued functions on a totally disconnected compact Hausdorff space.*

Proof. This follows from the preceding theorem. It is known that B can be represented as the algebra of open and closed sets in its representation space, and in this case the n -valued B -measurable functions are exactly the continuous n -valued functions.

In particular, e_i corresponds to the constant function of value i , i

$= 0, 1, \dots, n-1$, and, if x corresponds to f , then $C_i(x)$ corresponds to the function $C_i(f)$ which has the value $n-1$ on those points of S for which $f=i$, and has the value 0 on the other points of S . This, together with the latter example of the preceding section, leads to the following theorem.

THEOREM 17. *The lattice L is a Post algebra of order n , and any Post algebra of order n is such a lattice.*

Proof. P. C. Rosenbloom [9] based his axiom system for Post algebras on the operation of "prime," which is defined for n -valued functions on a set S as follows. If f is such a function, then $f^{(')}$ is that function which, if f has the value i at $P \in S$, has the value $i+1$ at P if $i=0, 1, \dots, n-2$, and has the value 0 if f has the value $n-1$ at P .

The lattice of functions described in the example of the preceding section is clearly closed under the operation of prime, since

$$f^{(')} = e_1 C_0(f) \vee e_2 C_1(f) \vee \dots \vee e_{n-2} C_{n-3}(f) \vee C_{n-2}(f).$$

Conversely, a lattice of n -valued functions on a set S which is closed under the operation of prime must also include the constant functions e_i and the functions $C_i(f)$, $i=0, 1, \dots, n-1$. This follows from the easily verified formulas

$$e_i = \left(\bigvee_{j=0}^{n-1} f^{(j)} \right)^{(i+1)} \quad \text{for any } f,$$

and

$$C_i(f) = \left(\bigvee_{j \neq n-i-1} f^{(j)} \right)^{(')},$$

where $f^{(j)}$ is defined inductively by $f^{(j+1)} = (f^{(j)})^{(')}$.

Since P. C. Rosenbloom [9] and L. I. Wade [12] have shown that Rosenbloom's axioms for Post algebras of order n describe lattices of n -valued functions closed under prime, the stated equivalence then follows.

Hence results obtained so far are equally valid for Post algebras of order n . Thus, in particular, Theorem 15 led to the representation of Post algebras given by Theorem 16. The following theorem gives a further result along these lines.

THEOREM 18. *The lattice L is complete and atomistic if and only if L is isomorphic with the set of all n -valued functions on a set S .*

Proof. It is clear from Theorem 16 that the set E of elements less than or equal to e_1 is isomorphic with B under the correspondence $b \rightarrow e_1 b$. If L is complete and atomistic, then E is complete and atomistic, and hence is isomorphic with the class of all subsets of the set of all atoms of E . Hence B is

isomorphic with the class of all subsets of a set S , every n -valued function on S is B -measurable, and the result follows by Theorem 15. The converse is obvious.

In what follows, all prime ideals are nonzero and proper.

THEOREM 19. *Every maximal chain of properly ascending prime ideals in L consists of exactly $n-1$ prime ideals, and every prime ideal is a member of exactly one such chain.*

Proof. The proof of Kaplansky's Theorem [3] shows that every prime ideal of the representation space given by Theorem 16 is associated with exactly one point of the representation space. That is, if S denotes now the totally disconnected compact Hausdorff space, P is a prime ideal of L , and f and g are continuous n -valued functions on S , then there exists exactly one point $p \in S$ with the property that if f is a member of P and if $g \leq f$ in some neighborhood N_p of p , then g is also a member of P . However $g \leq f$ at p if and only if $g \leq f$ in some neighborhood N_p , for if $g(p) = i$ and $f(p) = j$, let $N_p = g^{-1}(i) \cap f^{-1}(j)$, and the converse is obvious. Hence every prime ideal is of the form $P_{pi} = \{f | f(p) \leq i-1\}$, where $i = 1, 2, \dots, n-1$ and $p \in S$. Moreover, if p and p_0 are different points of S , then there exists $b \in B$ with $p_0 \notin \lambda(b)$ and $p \in \lambda(b)$, where $\lambda(b)$ is the open and closed subset of S which is the image of b . Therefore if b corresponds to the function f , then $f(p) = n-1$ and $f(p_0) = 0$, since $p \in \lambda(C_{n-1}(b)) = \lambda(b)$ and $p_0 \in \lambda(C_0(b)) = \lambda(b)$. Hence $P_{p_0i} \not\subseteq P_{pj}$ for $p \neq p_0$ and any $i = 1, 2, \dots, n-1$ and $j = 1, 2, \dots, n-1$. This completes the proof.

It follows from this argument that

$$\{e_i\} = \bigcap_{p \in S} (P_{p(i+1)} - P_{pi}), \quad i = 0, 1, \dots, n-1,$$

where P_{p_0} stands for the empty set and P_{pn} stands for the entire set of functions in the representation space. For it is clear that $e_i \in P_{p(i+1)} - P_{pi}$ for each $p \in S$, $i = 0, 1, \dots, n-1$. Now, for any $i(p)$, suppose that

$$f \in \bigcap_{p \in S} (P_{p(i(p)+1)} - P_{pi(p)}) \quad \text{and} \quad g \in \bigcap_{p \in S} (P_{p(i(p)+1)} - P_{pi(p)}).$$

Then, if $f \not\leq g$, there exists a prime ideal P which contains g but not f . By Theorem 19 the prime ideal P is of the form $P = P_{p_0i_0}$ for some $p_0 \in S$ and some $i_0 = 1, 2, \dots, n-1$. This contradicts

$$f \in P_{p_0(i(p_0)+1)} - P_{p_0i(p_0)} \quad \text{and} \quad g \in P_{p_0(i(p_0)+1)} - P_{p_0i(p_0)}.$$

Hence there is at most one element in $\bigcap_{p \in S} (P_{p(i(p)+1)} - P_{pi(p)})$ for any $i(p)$. Thus, in particular, for the constant values of $i(p)$, $\{e_i\} = \bigcap_{p \in S} (P_{p(i+1)} - P_{pi})$ for each $i = 0, 1, \dots, n-1$.

These remarks lead to the following characterization of finite Post algebras of order n .

THEOREM 20. *Let L_D be a finite distributive lattice in which every maximal chain of properly ascending prime ideals consists of exactly $n-1$ prime ideals, and every prime ideal is a member of exactly one such chain. Then L_D is a Post algebra of order n .*

Proof. Denote the chains of prime ideals by $P_{p_1} \subset P_{p_2} \subset \cdots \subset P_{p_{(n-1)}}$, where p ranges over some index set S . Let P_{p_0} denote the empty set \emptyset and P_{p_n} denote the entire set L_D . Then for every prime ideal $P \subset L_D$ there exists one and only one $p \in S$ and i such that $P = P_{p_i}$.

For any $x \in L_D$ and any $p \in S$ there exists $i(p, x) = i$ such that

$$x \in P_{p(i+1)} - P_{p_i}.$$

Hence $x \in \bigcap_{p \in S} (P_{p(i(p,x)+1)} - P_{p_i(p,x)})$. Since such an intersection has at most one point, by the remarks preceding this theorem,

$$\{x\} = \bigcap_{p \in S} (P_{p(i+1)} - P_{p_i}).$$

If $y \in L_D$ and $j(p, y) = j$, so that $\{y\} = \bigcap_{p \in S} (P_{p(j+1)} - P_{p_j})$, it is easy to see that

$$(1) \quad \{x \vee y\} = \bigcap_{p \in S} (P_{p(\max(i,j)+1)} - P_{p(\max(i,j))}),$$

and

$$(2) \quad \{xy\} = \bigcap_{p \in S} (P_{p(\min(i,j)+1)} - P_{p(\min(i,j))}),$$

since the element at the left is contained in each of the sets in the intersection at the right.

On the other hand, if L_D is finite, then for any $i(p)$, $\bigcap_{p \in S} (P_{p(i(p)+1)} - P_{p_i(p)}) \neq \emptyset$. To prove this, note the following lemma.

LEMMA 4. *If $p_1 \neq p_2$, $p_1 \in S$ and $p_2 \in S$, then $(P_{p_1(i+1)} - P_{p_1 i}) \cap P_{p_2 k} \neq \emptyset$ for any $i = 0, 1, \dots, n-1$ and any $k = 1, 2, \dots, n$.*

Proof. If $P_{p_1(i+1)} - P_{p_1 i} \subset L - P_{p_2 k}$, then taking the complement of both sides, $P_{p_2 k} \subset P_{p_1 i} \cup (L - P_{p_1(i+1)})$, so that

$$P_{p_1(i+1)} \cap P_{p_2 k} \subset P_{p_1(i+1)} \cap (P_{p_1 i} \cup (L - P_{p_1(i+1)})) = P_{p_1 i}.$$

Hence $P_{p_1 i} \supset P_{p_1(i+1)}$ or $P_{p_1 i} \supset P_{p_2 k}$, contradicting the hypotheses of the theorem.

Now let S_m be a set of cardinality m , where $S_m \subset S$. Since L_D is finite, S itself is finite. Now for any $i(p) = i$ and any S_m , $\bigcap_{p \in S_m} (P_{p(i+1)} - P_{p_i}) \neq \emptyset$. It follows easily from Lemma 4 that this statement is true for $m = 2$, so suppose it is true for $m = k$. The set S_{k+1} has $k+1$ distinct subsets of the form S_k , S_{k1} , S_{k2} , \dots , $S_{k(k+1)}$, each of these being obtained by deleting a particular point of S_{k+1} . Let $x_j \in \bigcap_{p \in S_{kj}} (P_{p(i+1)} - P_{p_i})$ and let $\omega_k = \bigvee_{m=1}^{k+1} (\bigwedge_{j \neq m} x_j)$. Then

it is straightforward to verify that $\omega_k \in \bigcap_{p \in S_{k+1}} (P_{p(i+1)} - P_{pi})$. Hence $\bigcap_{p \in S} (P_{p(i(p)+1)} - P_{pi(p)}) \neq \emptyset$ for any $i(p)$.

The theorem itself now follows, for it is easy to verify, using (1) and (2), that Axioms 1, 2, and 3 are satisfied, where

$$\{e_i\} = \bigcap_{p \in S} (P_{p(i+1)} - P_{pi}),$$

and

$$\{C_j(x)\} = \bigcap_{x \in P_{p(j+1)} - P_{pj}} (L - P_{p(n-1)}) \cap \bigcap_{x \in P_{p(j+1)} - P_{pj}} P_{p1}.$$

In the case where L_D is infinite, it need not be true that

$$\bigcap_{p \in S} (P_{p(i(p)+1)} - P_{pi(p)}) \neq \emptyset$$

for arbitrary $i(p)$, where now S also is infinite. C. C. Chang and A. Horn have proved a corresponding theorem for the case of an infinite lattice, the additional assumption being a hypothesis which requires the existence of the e_i , $i=0, 1, \dots, n-1$.

7. Completeness properties. This section is devoted to certain results on infinite Post algebras. The proofs are dependent, in large part, on the formulation in §2.

THEOREM 21. *If $b_i \in B$ and either of the quantities*

$$\bigvee_{B; i \in I} b_i, \quad \bigvee_{L; i \in I} b_i$$

exists, then the other exists and the two are equal.

Proof. If $b = \bigvee_{B; i \in I} b_i$ exists let $y \in L$ and $y \geq b_i$ for all $i \in I$. Then $C_{n-1}(y) \geq C_{n-1}(b_i) = b_i$ for all $i \in I$ by Theorem 9 and the Corollary of Theorem 3. Hence $C_{n-1}(y) \geq b$, and $y \geq C_{n-1}(y) \geq b$.

Conversely, if $r = \bigvee_{L; i \in I} b_i$ exists, then likewise $C_{n-1}(r) \geq C_{n-1}(b_i) = b_i$ for all $i \in I$. Hence $C_{n-1}(r) \geq r$. Since $r \geq C_{n-1}(r)$, $r = C_{n-1}(r) \in B$.

COROLLARY. *If $b_i \in B$ and either of the quantities*

$$\bigwedge_{B; i \in I} b_i, \quad \bigwedge_{L; i \in I} b_i$$

exists, then the other exists and the two are equal.

Proof. This follows from the duality theorem.

THEOREM 22. *If $x = \bigvee_{L; i \in I} x_i$ exists, where $x_i \in L$ for all $i \in I$, then $\bigvee_{B; i \in I} D_j(x_i)$ exists and $D_j(x) = \bigvee_{B; i \in I} D_j(x_i)$ for each $j=1, 2, \dots, n-1$. Conversely, if $\bigvee_{B; i \in I} D_j(x_i)$ exists for each $j=1, 2, \dots, n-1$, where $x_i \in L$ for all $i \in I$, then*

$\bigvee_{L; i \in I} x_i$ exists and is equal to the element x which is determined by the formulas $D_j(x) = \bigvee_{B; i \in I} D_j(x_i)$. The dual statements are also true.

Proof. If $x = \bigvee_{L; i \in I} x_i$ exists, then $x \geq x_i$ for all $i \in I$, so that $D_j(x) \geq D_j(x_i)$ for all $i \in I$ and each $j = 1, 2, \dots, n-1$, by Theorem 9. Secondly, suppose that $w \geq D_j(x_i)$ for some fixed $j = 1, 2, \dots, n-1$ and all $i \in I$. Let $w' = C_{n-1}(w) \vee e_{j-1}$. Then it follows from Theorem 12, and from Theorem 3 and Theorem 4, that $C_{n-1}(w') = C_{n-1}(w)$, and also that $C_i(w') = 0$ for $i = j, j+1, \dots, n-2$. Hence

$$D_j(w') = \bigvee_{i=j}^{n-1} C_i(w') = C_{n-1}(w).$$

It is clear from Axiom 3 that, for each $i \in I$, $x_i \leq e_{j-1} \vee D_j(x_i)$. Since $w \geq D_j(x_i)$, $C_{n-1}(w) \geq C_{n-1}(D_j(x_i)) = D_j(x_i)$ by Theorem 9 and Theorem 3. Hence $x_i \leq e_{j-1} \vee C_{n-1}(w) = w'$ for all $i \in I$. Thus $x \leq w'$, and $D_j(x) \leq D_j(w') = C_{n-1}(w) \leq w$.

For the converse $\bigvee_{B; i \in I} D_k(x_i) \geq \bigvee_{B; i \in I} D_j(x_i) \in B$ for all j and k in the range $1 \leq k \leq j \leq n-1$, so that

$$\bigvee_{B; i \in I} D_j(x_i) = D_j(x), \quad \text{where} \quad x = \bigvee_{j=1}^{n-1} e_j \left(\bigvee_{B; i \in I} D_j(x_i) \right),$$

by Theorem 7. Hence $D_j(x) \geq D_j(x_i)$ for all $i \in I$ and each $j = 1, 2, \dots, n-1$, and by Theorem 9, $x \geq x_i$ for all $i \in I$. If $y \geq x_i$ for all $i \in I$, then $D_j(y) \geq D_j(x_i)$ for all $i \in I$ and each $j = 1, 2, \dots, n-1$, so that $D_j(y) \geq D_j(x)$ for each $j = 1, 2, \dots, n-1$, and by Theorem 9 again, $y \geq x$.

The dual statements are equally valid. This follows from the duality theorem, where it is shown that $D'_j(x) = D_{n-j}(x)$ for each $j = 1, 2, \dots, n-1$.

THEOREM 23. *The lattice L is complete if and only if B is complete.*

Proof. If L is complete then B is complete by Theorem 21. If B is complete then L is complete by Theorem 22.

The infinite distributive law $y \bigvee_{i \in I} x_i = \bigvee_{i \in I} yx_i$ is valid in any Boolean algebra whenever $\bigvee_{i \in I} x_i$ exists. This law is also valid in any complete pseudo-complemented lattice. Hence, by Theorem 5, this law (and its dual) is valid in L whenever L is complete. That is, if the Post algebra L is complete, then L is a topological lattice [1, p. 146]. However the next theorem shows that this infinite distributivity law holds in any Post algebra L .

THEOREM 24. *The infinite distributive law $y \bigvee_{i \in I} x_i = \bigvee_{i \in I} yx_i$ is valid in any Post algebra L whenever $x = \bigvee_{i \in I} x_i$ exists. The dual statement is also true.*

Proof. By Theorem 22, for each $j = 1, 2, \dots, n-1$, $D_j(x) = \bigvee_{B; i \in I} D_j(x_i)$.

Thus

$$D_j(y)D_j(x) = D_j(y) \bigvee_{B, i \in I} D_j(x_i) = \bigvee_{B, i \in I} D_j(y)D_j(x_i),$$

since the law is valid in any Boolean algebra. Hence, by Theorem 11, $D_j(yx) = \bigvee_{B, i \in I} D_j(yx_i)$ for each $j=1, \dots, n-1$, and $yx = \bigvee_{L, i \in I} yx_i$ by Theorem 22. The duality theorem yields the dual result.

THEOREM 25. *If $x = \bigvee_{i \in I} x_i$ exists, then the following generalized rules of simplification, stated in recursive form, are valid.*

$$C_{n-1}(x) = \bigvee_{i \in I} C_{n-1}(x_i).$$

$$(1) \quad C_k(x) = \bigvee_{i \in I} \left(\bigwedge_{j=k+1}^{n-1} \{C_j(x)\}^- \right) C_k(x_i) \quad \text{for } k = n-2, n-3, \dots, 1, 0.$$

Dually, if $y = \bigwedge_{i \in I} y_i$ exists, then

$$C_0(y) = \bigvee_{i \in I} C_0(y_i).$$

$$(2) \quad C_k(y) = \bigvee_{i \in I} \left(\bigwedge_{j=0}^{k-1} \{C_j(y)\}^- \right) C_k(y_i), \quad \text{for } k = 1, 2, \dots, n-2, n-1.$$

Proof. The first statement follows from Theorem 6 and Theorem 22. Specifically, $C_{n-1}(x) = D_{n-1}(x)$, and for $k < n-1$,

$$\begin{aligned} C_k(x) &= D_k(x) \{D_{k+1}(x)\}^- \\ &= \left(\bigvee_{i \in I} D_k(x_i) \right) \left\{ \bigvee_{i \in I} D_{k+1}(x_i) \right\}^- \\ &= \left(\bigvee_{i \in I} D_k(x_i) \right) \left(\bigwedge_{i \in I} \{D_{k+1}(x_i)\}^- \right). \end{aligned}$$

Hence, by the infinite distributivity law,

$$\begin{aligned} C_k(x) &= \bigvee_{i \in I} \left(D_k(x_i) \bigwedge_{i \in I} \{D_{k+1}(x_i)\}^- \right) \\ &= \bigvee_{i \in I} \left(C_k(x_i) \bigwedge_{i \in I} \{D_{k+1}(x_i)\}^- \right) \\ &= \bigvee_{i \in I} C_k(x_i) \{D_{k+1}(x)\}^- \\ &= \bigvee_{i \in I} \left(C_k(x_i) \bigwedge_{j=k+1}^{n-1} \{C_j(x)\}^- \right). \end{aligned}$$

The second statement follows by the duality theorem, where it is shown that $C'_i(x) = \{C_{n-i-1}(x)\}^-$. Applying this to (1) yields

$$\begin{aligned}\{C_0(y)\}^- &= \bigwedge_{i \in I} \{C_0(y_i)\}^-, \\ \{C_{n-k-1}(y)\}^- &= \bigwedge_{i \in I} \left(\bigvee_{j=k+1}^{n-1} C_{n-j-1}(y) \vee \{C_{n-k-1}(y_i)\}^- \right) \\ &\quad \text{for } k = n-2, n-3, \dots, 1, 0.\end{aligned}$$

Then (2) follows from this by taking complements.

THEOREM 26. *The normal completion [1, p. 58] L_N of a Post algebra is also a Post algebra. Furthermore B_N , the normal completion of B , is the Boolean algebra of complemented elements of L_N .*

Proof. The lattice L can be represented as the set of n -valued B -measurable functions on the representation space of B ; that is, the set of functions f such that $f^{-1}(i)$ for each $i=0, 1, \dots, n-1$ is an open and closed set. It is well known that B_N is isomorphic with the algebra of regular open sets in the representation space of B [1, p. 177]. Let L_N be the set of n -valued B_N -measurable functions on the representation space of B ; that is, the set of functions f such that $f^{-1}(i)$ for each $i=0, 1, \dots, n-1$ is a regular open set. Then $L \subset L_N$ and, since B_N is complete, L_N is complete by Theorem 23.

Now for any $x \in L_N$ it will be shown that there exist $x_i \in L$, $i \in I$, such that $x = \bigvee_{i \in I} x_i$. Then by the duality theorem it will follow that there exist $y_i \in L$, $i \in I$, such that $x = \bigwedge_{i \in I} y_i$, and this will complete the proof that L_N is the normal completion of L . If $x \in L_N$, there exist $C_j(x) \in B_N$, where $j=0, 1, \dots, n-1$, with $x = \bigvee_{j=1}^{n-1} e_j C_j(x)$. Since B_N is the normal completion of B , there exist $x_{ji} \in B$ such that $\bigvee_{B_N; i \in I} x_{ji} = C_j(x) \in B_N$, for each $j=1, 2, \dots, n-1$. For each i the quantities $x_{(n-1)i}, \dots, x_{2i}, x_{1i}, \bigwedge_{j=1}^{n-1} \bar{x}_{ji}$ are pairwise disjoint and their supremum is u . Hence, by the uniqueness theorem, if $x_i = \bigvee_{j=1}^{n-1} e_j x_{ji}$, then $C_j(x_i) = x_{ji}$ for each $j=1, 2, \dots, n-1$ and all $i \in I$. Now each of the quantities $\bigvee_{B_N; i \in I} D_j(x_i)$ exist for $j=1, 2, \dots, n-1$, so that Theorem 22 yields $x = \bigvee_{L_N; i \in I} x_i$.

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